Homomorphisms and Short Exact Sequences of Skew Bracoids

Isabel Martin-Lyons

joint work with Paul Truman

Keele University, UK

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Outline

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The Objects at Play

Definition

A *skew (left) brace* is a triple $(G, \star, \circ)$, where $(G, \star)$ and $(G, \circ)$ are groups and for all $g, h, f \in G$

$$g \circ (h \star f) = (g \circ h) \star g^{-1} \star (g \circ f).$$

Definition

A *skew (left) bracoid* is a 5-tuple $(G, \cdot, N, \star, \odot)$, where $(G, \cdot)$ and $(N, \star)$ are groups and $\odot$ is a transitive action of G on N for which

$$g \odot (\eta \star \mu) = (g \odot \eta) \star (g \odot e_N)^{-1} \star (g \odot \mu),$$

for all $g \in G$ and $\eta, \mu \in N$.

Housekeeping

- We will assume everything is finite.
- We will frequently write $(G, N, \odot)$, or even (G, N), for $(G, \cdot, N, \star, \odot)$.
- We will refer to $(N, \star)$ as the additive group and $(G, \cdot)$ as the multiplicative or acting group.
- Any identity will be denoted e, possibly with a subscript.

For example

Examples

- Any skew brace $(G, \star, \circ)$ can be thought of as a skew bracoid $(G, \circ, G, \star, \odot)$, where $\odot$ is simply $\circ$. If (G, N) is a skew bracoid with $|G| = |N|$ we say that (G, N) is *essentially a skew brace*.
- For any group (G) we have the skew bracoid $(G, \{e\}, \odot)$ where of course the action $\odot$ is trivial.
- Let $d, n \in \mathbb{N}$ such that $d|n$. Take $G = \langle r, s \mid r^n = s^2 = e, srs^{-1} = r^{-1} \rangle \cong D_n$ and $N = \langle \eta \rangle \cong C_d$. Then we get a skew bracoid $(G, N, \odot)$ for $\odot$ given by

$$r^i s^j \odot \eta^k = \eta^{i+(-1)^j k}.$$

The γ-functions

Definition/Proposition

Let $(G, N, \odot)$ be a skew bracoid and $g \in G$. The map $\gamma_g : N \to N$ given by

$$\gamma_g(\eta) = (g \odot e_N)^{-1}(g \odot \eta)$$

is in fact an automorphism of N.

We call these maps associated with the skew bracoid the γ-*functions* of the skew bracoid.

We can use these maps to embed G into $\mathrm{Hol}(N)$ via $g \mapsto (g \odot e_N, \gamma_g)$. Morally, this amounts to killing off any kernel of the action $\odot$.

Substructures

Let $(G, N, \odot)$ be a skew bracoid.

Definition

The triple $(H, M, \odot)$ is a *sub-skew bracoid* of $(G, N, \odot)$ if and only if

- H is a subgroup of G,
- M is a subgroup of N,
- and H acts transitively on M via $\odot$.

Definition

A normal subgroup M of N is an *ideal* of $(G, N, \odot)$ if and only if it is closed under γ_g for all $g \in G$.

Quotients

Proposition

Let $(G, N, \odot)$ be a skew bracoid with M an ideal. We have that G acts on the quotient group N/M via $g \odot (\eta M) := (g \odot \eta)M$, and $(G, N/M, \odot)$ is a skew bracoid.

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Homomorphisms

Definition

A *homomorphism* of skew bracoids between $(G, N, \odot)$ and $(G', N', \odot')$ is a pair of group homomorphisms $\varphi : G \to G'$ and $\psi : N \to N'$ such that

$$\psi(g \odot \eta) = \varphi(g) \odot' \psi(\eta)$$

for all $g \in G$ and $\eta \in N$.

An alternative framing

Let $\varphi : G \to G'$ be a homomorphism of groups.

If $\varphi(\mathrm{Stab}_G(e_N)) \subseteq \mathrm{Stab}_{G'}(e_{N'})$ then we have a well-defined map $\varphi_N : N \to N'$ given by

$$\varphi_N(g \odot e_N) := \varphi(g) \odot' e_{N'}$$

for all $g \in G$.

If this φ_N turns out to be a homomorphism itself then the pair, φ and φ_N, forms a homomorphism of skew bracoids.

Conversely every homomorphism of skew bracoids is necessarily of this form.

Notation

With this in mind, we set the following convention:

- $\boldsymbol{\varphi}$ denotes the pair of homomorphisms that form the skew bracoid homomorphism,
- φ denotes the homomorphism between the acting groups,
- and φ_N the (induced) homomorphism between the additive groups.

Isomorphism and Equivalence

Definition

An *isomorphism* of skew bracoids is a homomorphism of skew bracoids φ in which both φ and φ_N are isomorphisms of groups.

Definition

We say that two skew bracoids $(G, N, \odot)$ and $(G', N', \odot')$ are equivalent if and only if $N = N'$ and the image of G and the image of G' in $\mathrm{Hol}(N)$ coincide.

Injectivity and Surjectivity

The injectivity and surjectivity of the homomorphism on the acting group and the homomorphism on the additive group do not necessarily align.

As with ideals, it is the situation in the additive groups that we care about.

Definition

Let φ be a homomorphism of skew bracoids. We say that φ is *injective* (resp. *surjective*) if and only if φ_N is injective (resp. surjective).

This has the uncomfortable consequence that a homomorphism can be injective and surjective, but not an isomorphism. We can take comfort in the fact that it will be an isomorphism, up to our notion of equivalence.

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Images and Kernels

Proposition

Let $\varphi : (G, N, \odot) \to (G', N', \odot')$ be a homomorphism of skew bracoids then

- $\ker(\varphi_N)$ is an ideal of $(G, N, \odot)$,
- and $(\mathrm{im}(\varphi), \mathrm{im}(\varphi_N), \odot')$ is a sub-skew bracoid of $(G', N', \odot')$.

Definition

The *image* of the homomorphism φ is $\mathrm{im}(\varphi) := (\mathrm{im}(\varphi), \mathrm{im}(\varphi_N))$.

Kernel For Our Purposes

Remember our goal is to develop a notion of short exact sequence for skew bracoids. For there to be any hope of the kernel of a homomorphism and the image of a homomorphism aligning, we need them to be the same kind of object.

Naive Attempt: $(\ker(\varphi), \ker(\varphi_N))$

Consider the (toy) example $\boldsymbol{\varphi} : (D_3, C_3) \to (D_3, \{e\})$ where φ is just the identity, and $\varphi_N(\eta) = e$ for all $\eta \in C_3$.

But $\ker(\varphi) = \{e\}$ and $\ker(\varphi_N) = C_3$, so $\ker(\varphi)$ does not act transitively on $\ker(\varphi_N)$ and thus we so not have a skew bracoid.

For Our Purposes

Proposition (Intelligent Attempt)

Let $\varphi : (G, N, \odot) \to (G', N', \odot')$ be a homomorphism of skew bracoids and $S' = \mathrm{Stab}_{G'}(e_{N'})$. The triple $\ker(\varphi) := (\varphi^{-1}(S'), \ker(\varphi_N), \odot)$ is a sub-skew bracoid of $(G, N, \odot)$.

- In our toy example $S' = D_3$ and $\varphi^{-1}(D_3) = D_3$ so that our "kernel" is (D_3, C_3). Certainly a skew bracoid.
- Specialising to the skew brace case, this stabiliser will necessarily be trivial so its inverse image is precisely the kernel of φ.
- This lands on the associated subgroup in G of $\ker(\varphi_N)$, our standard lift from an ideal in the additive group to subgroup of the acting group.

Short Exact Sequence

Then a short exact sequence of skew bracoids is given by

$$e \longrightarrow (G_1, N_1) \overset{\varphi}{\hookrightarrow} (G_2, N_2) \overset{\psi}{\twoheadrightarrow} (G_3, N_3) \longrightarrow e$$

where φ and ψ are homomorphisms of skew bracoids such that

$$\mathrm{im}(\varphi) = \ker(\psi).$$

Outline

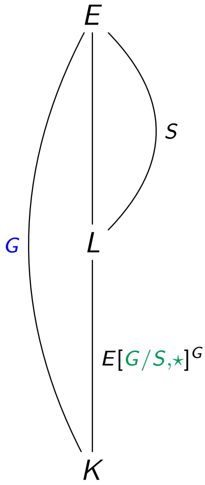
The Correspondence between Hopf-Galois Structures and Skew Bracoids

Let L/K be a separable extension of fields with Galois closure E, and write $(G, \cdot) = \mathrm{Gal}(E/K)$ and $S = \mathrm{Gal}(E/L)$. Recall

Theorem

There is a bijective correspondence between

- Hopf-Galois structures on L/K and
- operations $\star$ such that $(G, \cdot, G/S, \star, \odot)$ forms a skew bracoid, where $\odot$ is left translation of cosets via $\cdot$.



Almost Classical Hopf-Galois Structures

Definition

We say that L/K is *almost classically Galois* if S has a normal complement H in G.

Definition

A Hopf-Galois structure is *almost classical* if it corresponds under Greither-Pareigis to a subgroup of $\mathrm{Perm}(G/S)$ of the form $\lambda(H)^{opp}$, for some normal complement H of S.

Almost Classical Skew Bracoids

Definition

Let (G, N) be a skew bracoid and $S = \mathrm{Stab}_G(e_N)$. We say (G, N) is *almost classical* if S has a normal complement H in G for which (H, N) is essentially a trivial skew brace.

This means that for all $h_1, h_2 \in H$ we have

$$h_1 h_2 \odot e_N = (h_1 \odot e_N)(h_2 \odot e_N).$$

For Example

Example

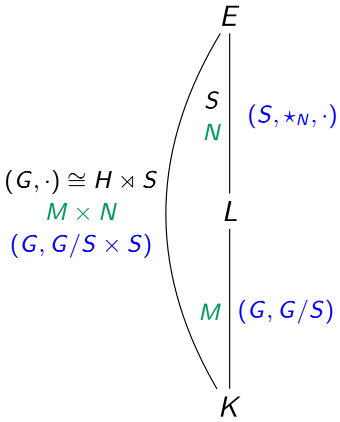
We can take $(G, N, \odot)$ with $G = \langle r, s \mid r^3 = s^2 = e, srs^{-1} = r^{-1} \rangle \cong D_3$, $N = \langle \eta \rangle \cong C_3$, with $\odot$ given by

$$r^i s^j \odot \eta^k = \eta^{i+(-1)^j k}. \tag{1}$$

- Putting $k = 0$ in (1), we get $r^i s^j \odot e_N = \eta^i$. We stabilise e_N precisely when $i = 0$, so $\mathrm{Stab}_G(e_N) = \langle s \rangle$.
- For $0 \leq i, j \leq 2$, we have $(r^i \odot e)(r^j \odot e) = \eta^{i+j} = r^i r^j \odot e$.
- $G \cong \langle r \rangle \rtimes \langle s \rangle$.

Hence we have an almost classical skew bracoid.

Induced Hopf-Galois Structures



Here $N \subseteq \mathrm{Perm}(S)$ and $M \subseteq \mathrm{Perm}(G/S)$, leading to a subgroup of $\mathrm{Perm}(G)$ isomorphic to $M \times N$.

Taking a direct product of the additive groups of $(G, \cdot, G/S, \star_M, \odot)$ and $(S, \star_N, \cdot)$ we can construct a skew bracoid $(G, G/S \times S)$ on E/K, which is essentially a skew brace.

For Example

Example

Again take $(G, \cdot) = \langle r, s \rangle \cong D_3$ and $(S, \cdot) = \langle s \rangle \cong C_2$.

We have the trivial skew brace $(S, \cdot, \cdot)$ and the almost classical skew bracoid $(G, G/S)$ with $G/S \cong \langle r \rangle$. Taking the direct product of G/S and S we get a cyclic group of order 6 generated by (rS, s) for example. Associating (rS, e) with r and (eS, s) with s, we get a skew brace $(G, \star, \cdot)$ where $(G, \star) \cong C_6$ and $(G, \cdot) \cong D_3$.

The Short Exact Sequence

With this machinery in tow we have the following.

Proposition

Let (G, H) be an almost classical skew bracoid and $S = \mathrm{Stab}_G(e_H)$. We have the short exact sequence

$$e \longrightarrow (S, S) \hookrightarrow (G, H \times S) \twoheadrightarrow (G, H) \longrightarrow e.$$

Thank you for your attention!